Plan: lecture 1: Hamiltonian ODEs, B. I. M. argument for ODEs.

lecture 2: ∞ dimension.

lecture 3: Quasi - invariance

Hamiltonian ODEs

Let $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be smooth, and consider the system

$$\begin{cases} \dot q = \dfrac{\partial H(p,q)}{\partial p}, \\[2mm] \dot p = -\dfrac{\partial H(p,q)}{\partial q}. \end{cases}$$

For instance, $A \geq 0$, $\quad H(p,q) = \frac{1}{2}|p|^2 + \frac{1}{2}|Aq|^2 + \frac{1}{p}|q|^p$

which leads to

$$\begin{cases} \dot q = p, \\ \dot p = -Aq - |q|^{p-2} q \end{cases}$$

consider $1 < p < 2$

problems when $q = 0$!

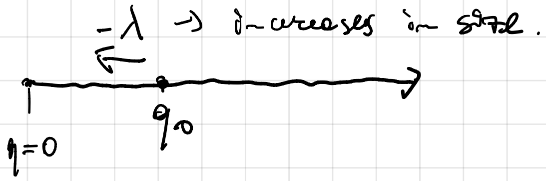
By standard Cauchy theory, we know that $\forall (p,q)$ with $q \neq 0$, $\exists$ $T_* > 0$ s.t. solution $\exists !$. Moreover,

$$T_* \geq c \left(|q|^{2-p} \wedge 1 \right).$$

Q: Do P ever hit $q = 0$? In general, yes

$$\frac{d}{dt}\left(\frac{1}{2}|Aq|^2 + \frac{1}{p}|q|^p \right) = -p \cdot \dot p = \left(Aq + |q|^{p-2} q\right) p$$

So if $A = id$, $q_0 = e_1$, $p_0 = -\lambda e_1$, then everything is 1d,



Claim: This is really rare - it happens for a set of Leb. measure 0 (when $d \geq 2$).

Sub-invariant measures

Def.: Let X be a Polish space (metric, complete, separable), and suppose that $\infty \in X$.

We say that $\Phi : \mathbb{R}_{\geq 0} \times X \to X$ is a Borel flow if

1. $\Phi(\cdot, x) : \mathbb{R}_{\geq 0} \to X$ is continuous,
2. $\Phi(t, \infty) = \infty \quad \forall t \geq 0$,
3. $\forall t, s \geq 0, \forall x \in X, \; \Phi(t+s, x) = \Phi(t, \Phi(s, x))$
4. $\Phi(0, x) = x \quad \forall x \in X$,
5. $\Phi(t, \cdot) : X \to X$ is a Borel-meas. map.

Def.: Let μ be a Borel measure on $X \setminus \{\infty\}$, and suppose that we have a Borel flow Φ. We say that μ is sub-invariant for the flow if for every $A \subseteq X \setminus \{\infty\}$ Borel, $\forall t \geq 0$,

$$\mu(\{\Phi(t, x) \in A\}) \leq \mu(A).$$

Equivalently,
- $\mathbb{1}_{X \setminus \{\infty\}} (\Phi(t, \cdot))_{\#} \mu \leq \mu$
- $\forall F : X \setminus \{\infty\} \to \mathbb{R}$, $F \geq 0$, Borel-meas.,
- $\int F(\Phi(t, x)) \, d\mu(x) \leq \int F(x) \, d\mu(x).$

If $=$ instead of $\leq$, we say that μ is invariant.

How do we find (sub)-invariant measures?

Let $X = \mathbb{R}^d \times \mathbb{R}^d \cup \{\infty\}$, suppose $\mu = \exp(-E(p,q))\, dp\, dq$. (I.3)

What is

$$\frac{d}{dt} \int F(p(t), q(t)) \exp(-E(p,q))\, dp\, dq \Big|_{t=0}$$

$$= \int \nabla F(q(t), p(t)) \cdot \begin{pmatrix} \dot q \\ \dot p \end{pmatrix} \exp(-E(p,q))\, dp\, dq \Big|_{t=0}$$

$$= \int (\partial_q F \cdot \partial_p H - \partial_p F \cdot \partial_q H) \cdot \exp(-E(p,q))\, dp\, dq$$

$$\overset{IBP}{=} - \int F \Big(\underbrace{\sum \partial_{p_i} \partial_{q_i} H - \partial_{q_i} \partial_{p_i} H}_{0} \Big) \exp(-E(p,q))\, dp\, dq$$

$$+ \int F (\partial_p H \cdot \partial_q E - \partial_q H \cdot \partial_p E) \exp(-E(p,q))\, dp\, dq.$$

Examples: $E = 0 \rightsquigarrow$ Leb. measure OK

$E = H \rightsquigarrow$ Gibbs measure OK.

In general, if $\frac{d}{dt} E(q(t), p(t)) \Big|_{t=0} = 0$,

we get sub-invariance. (where $\infty = \{H$ is not smooth$\}$).

How does this help? Consider again the flow above.

$\mu = \exp(-H(q,p))\, dq\, dp$ is a finite measure on

$\mathbb{R}^d \times \mathbb{R}^d$, "Bad set" $\{0\} \times \mathbb{R}^d$. Define

$$X = \mathbb{R}^d \times \mathbb{R}^d / \{0\} \times \mathbb{R}^d, \qquad \infty := \{0\} \times \mathbb{R}^d.$$

(1.4)

Since $\mu(\{0\} \times \mathbb{R}^d) = 0$, then μ is also defined on X, and sub-invariant.

$$\mu(\{\Phi(1,x) = \infty\}) = \mu(\{\Phi(\tfrac{1}{2},x) = \infty\}) + \mu(\{\Phi(\tfrac{1}{2},\Phi(\tfrac{1}{2},x)) = \infty\} \cap \{\Phi(\tfrac{1}{2},x) \neq \infty\})$$

By sub-invariance, $\leq \mu(\{\Phi(\tfrac{1}{2},x) = \infty\}) + \mu(\{\Phi(\tfrac{1}{2},x) = \infty\})$.

<u>Thm:</u> (Bourgain's invariant measure argument)

Let $\Phi : X \to \mathbb{R}$ be a Borel flow. Let μ be sub-invariant, and suppose that

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \mu(\{\Phi(\varepsilon, x) = \infty\}) = 0.$$

Then $\mu(\{x \in X : \exists t \geq 0 \text{ s.t. } \underset{\overset{\shortparallel}{t}}{\Phi(t,x)}\}) = 0$. Moreover, if μ is finite, then μ is invariant.

<u>Pf:</u> Note that, $\forall\, t, \tau \geq 0$,

$$\{\Phi(t+\tau, x) = \infty\} = \{\Phi(\tau, \Phi(t,x)) = \infty\}$$

$$= \{\Phi(t,x) = \infty\} \cup (\{\Phi(\tau, \Phi(t,x)) = \infty\} \cap \{\Phi(t,x) \neq \infty\}).$$

Therefore,

2,5

$$\mu(\{\Phi(t+\tau, x) = \infty\}) \le \mu(\{\Phi(t,x) = \infty\}) + \mu(\{\Phi(\tau, x) = \infty\}).$$

Inductively, for $t > 0$,

$$\mu(\{\Phi(t,x) = \infty\}) \le \mu(\{\Phi(t-1/n, x) = \infty\}) + \mu(\{\Phi(t/n, x) = \infty\})$$

$$\le n\,\mu\left(\left\{\Phi\left(\tfrac{t}{n}, x\right) = \infty\right\}\right) + \overbrace{\cancel{\mu(\{\Phi(0,x) = \infty\})}}^{0}$$

$\to 0$ as $n \to \infty$. Therefore, $\mu(\{\Phi(t,x) = \infty\}) = 0$.

Since $E = \bigcup_{n=1}^{\infty} \{\Phi(n,x) = \infty\}$, then

$\mu(E) = 0$.

Invariance: suppose μ finite. Then, $\forall S \subseteq X \setminus \{\infty\}$ Borel,

$$\mu(\{\Phi(t,x) \in S\}) \le \mu(S) \quad \text{by sub-invariance.}$$

$$\mu(\{\Phi(t,x) \in S^c\}) = \mu(\{\Phi(t,x) \in S^c \setminus \{\infty\}\}) + \mu(\{\Phi(t,x) \in \{\infty\}\})$$

$$\le \mu(S^c \setminus \{\infty\}).$$

Therefore,

$$\mu(X) = \mu(\{\Phi(t,x) \in S\} \cup \{\Phi(t,x) \in S^c\})$$

I-6

$$= \mu\left(\{\Phi(t,x)\in S\}\right) + \mu\left(\{\Phi(t,x)\in S^c \setminus \{\infty\}\}\right)$$

$$\leq \mu(S) + \mu(S^c)$$

$$= \mu(X),$$

so $=$ must be true everywhere.

$\triangle$ $\Phi_t(x) = e^t x$, $\mu = dx$ is subinvariant, $\mu(\{\Phi_t = \infty\}) = 0$, but not inv.

Cor.: In the example above, $q(t) \neq 0$ for a.e. p_0, q_0, as long as $d \geq 2$.

Pf.: let $\mu = \exp(-H(q,p))\,dq\,dp$. We know that

$$\mu\left(\{\min_{0\leq\tau\leq t} |q(\tau)| = 0\}\right) \leq \mu(\{\underset{\substack{\uparrow \\ \text{B.F.P.} \\ \text{time}}}{\tau_*} \leq t\})$$

$$\leq \mu\left(\{c|q|^{2-p} \leq t\}\right)$$

$$\leq C_d\, t^{\frac{d}{2-p}}.$$ Therefore, if $\frac{d}{2-p} > 1$

$\iff d > 3-p,$

$\Leftarrow d \geq 2,$

We have that

$$t^{-1}\mu\left(\{\Phi(t,q,p) = \infty\}\right)$$

$$\leq t^{\frac{d}{2-p}-1} \to 0 \quad \text{as } t \to 0. \qquad \square$$

Afternoon: PDEs.

Bourgain '94: Consider $i\partial_t u = -\Delta u + |u|^4 u$.

Then, formally, write

$$H(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{6}\int |u|^6 = \frac{1}{2}\int |\nabla \operatorname{Re} u|^2 + \frac{1}{2}\int |\nabla \operatorname{Im} u|^2 + \frac{1}{6}\int \left((\operatorname{Re} u)^2 + (\operatorname{Im} u)^2\right)^3$$

Then
$$\frac{\partial H}{\partial \operatorname{Re} u} = -\Delta \operatorname{Re} u + |u|^4 \operatorname{Re} u$$
$$\frac{\partial H}{\partial \operatorname{Im} u} = -\Delta \operatorname{Im} u + |u|^4 \operatorname{Im} u.$$

So
$$\partial_t \operatorname{Re} u = \partial_t \operatorname{Im}(iu) = -\Delta \operatorname{Im} u + |u|^4 \operatorname{Im} u = \frac{\partial H}{\partial \operatorname{Im} u}$$
$$\partial_t \operatorname{Im} u = -\partial_t \operatorname{Re}(iu) = -\frac{\partial H}{\partial \operatorname{Re} u}.$$

NLS Hamiltonian. Can we do the same with

$$\rho \sim \exp(-H(u))\, d\operatorname{Re} u\, d\operatorname{Im} u$$
$$\sim \exp\left(-\frac{1}{2}\int |\nabla u|^2 - \frac{1}{6}\int |u|^6\right) d\operatorname{Re} u\, d\operatorname{Im} u\,?$$

Problem: Who is ρ ?

(2.0)

Thm: let $\mathcal{B}$ be an ∞-dim. Banach space.
There exists no σ-finite measure μ on the Borel sets of $\mathcal{B}$
s.t.

1. It is invariant by translations,
2. $\exists r > 0$ s.t. $0 < \mu(B(0,r)) < \infty$.

Idea: $\frac{1}{Z}\exp\left(-\frac{1}{2}\langle A^{-1}x, x\rangle\right) dx$ is Gaussian.
On $\mathbb{R}^d$, Z

let e_j be a basis of $\mathbb{R}^d$ s.t. $Ae_j = \lambda_j e_j$. Let

$g_j \sim N(0,1)$. Then let 2.1

$$\mu = \mathrm{Law}\left(\sum_{j=1}^{d} \sqrt{\lambda_j}\, g_j\, e_j =: X \right).$$

Then $$\mu = \left(\prod_{j=1}^{d} \frac{1}{\sqrt{2\pi\lambda_j}} \right) \exp\left(- \sum_{j=1}^{d} \frac{\langle x, e_j \rangle^2}{\lambda_j} \right) dx$$

Pf.: Let $$\mu_0(z) = \prod_{j=1}^{d} \frac{1}{\sqrt{2\pi}} \exp\left(-|z_j|^2/2 \right)$$
$$= \mathrm{Law}\begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} =: \mathrm{Law}(Z)$$

Let $$\tilde{A} z = \sum_{j=1}^{d} \sqrt{\lambda_j}\, z_j\, e_j.$$

Have that $$\int F(x)\, d\mu(x) = \mathbb{E}\left[F(X) \right]$$

$$= \mathbb{E}\left[F(\tilde{A} Z) \right] = \int F(\tilde{A} z)\, d\mu_0(z)$$

$$= \frac{1}{(2\pi)^{d/2}} \int F(\tilde{A} z) \exp\left(-\frac{1}{2} \sum |z_j|^2 \right) dz_j$$

$$= \frac{1}{(2\pi)^{d/2}} \int F(x) \exp\left(-\frac{1}{2} \sum \frac{\langle x, e_j \rangle^2}{\lambda_j} \right) \frac{\prod dx_j}{\det(\tilde{A})}$$

$$= \int F(x) \exp\left(-\frac{1}{2} \langle A^{-1} x, x \rangle \right) \prod \frac{1}{\sqrt{2\pi\lambda_j}}\, dx.$$

Idea: Take $d = \infty$, $A = (1-\Delta)^{-1}$, $\rho = \exp\left(-\frac{1}{6} \int |u|^6 \right) d\mu$

Work in Fourier basis, so A diagonal. (2.2)

$$e_n := e^{2\pi i n x}, \quad (1-\Delta)^{-1} e_n = \frac{1}{1+(2\pi n)^2} e_n =: \frac{1}{\langle n\rangle^2} e_n.$$

Then $\mu = \text{Law}\left(\sum_{-\infty}^{+\infty} \frac{g_n}{\langle n\rangle} e_n \right)$.

Here $g_n = a_n + i b_n$, where $a_n, b_n \sim N(0,1)$, i.i.d.

Since $\int |u|^6 \geq 0$, just need to show that

$\mathbb{E}\left[\int_{\mathbb{T}} |u|^6\right] < \infty$ to define the measure.

$$\mathbb{E} \int_{\mathbb{T}} |u|^6(x)\,dx = \int_{\mathbb{T}} \mathbb{E}\left[|u|^6(x)\right]$$

$$\leq C \int_{\mathbb{T}} \left(\mathbb{E}\left[|\mathrm{Re}\, u|^6_{(x)}\right] + \mathbb{E}\left[\mathrm{Im}\, u^6(x)\right]\right) dx$$

We know that $\mathrm{Re}\, u(x)$ is Gaussian, $\mathbb{E}\left[\mathrm{Re}\, u(x)\right] = 0$. Therefore,

$$\frac{\mathrm{Re}\, u(x)}{\left(\mathbb{E}|\mathrm{Re}\, u(x)|^2\right)^{1/2}} \sim N(0,1), \quad \text{and} \quad \mathbb{E}\left|\frac{\mathrm{Re}\, u(x)}{\sqrt{\mathbb{E}|\mathrm{Re}\, u(x)|^2}}\right|^6 = 15.$$

Fact: If G Gaussian (even ∞ dim.), and P polynomial of degree d, $p \geq 2$, then

$$\left(\mathbb{E}\,|P(G)|^p\right)^{1/p} \leq (p-1)^{d/2} \left(\mathbb{E}\,|P(G)|^2\right)^{1/2}.$$

$$\leq C_{p,\alpha} \left(\mathbb{E}\,|P(G)|^\alpha\right)^{1/\alpha} \quad \forall \alpha > 0.$$

Going back,

$$\mathbb{E}\,|u(x)|^2 = \mathbb{E}\left[\sum_{n\in\mathbb{Z}} \frac{g_n}{\langle n\rangle} e^{2\pi i n\cdot x} \sum_{m\in\mathbb{Z}} \frac{\overline{g_m}}{\langle m\rangle} e^{-2\pi i m\cdot x}\right]$$

$$= \sum_{n,m \in \mathbb{Z}} \frac{\mathbb{E}[g_n \overline{g_m}]}{\langle n \rangle \langle m \rangle} e^{2\pi i (n-m)\cdot x}$$

$$= \sum_{n,m \in \mathbb{Z}} \frac{2\,\delta_{nm}}{\langle n \rangle^2} = \sum_{n \in \mathbb{Z}} \frac{2}{\langle n \rangle^2} =: \sigma^2 < \infty .$$

Therefore, $\mathbb{E} \int_{\mathbb{T}} |v|^6 (x)\, dx \leq C\, \sigma^2 < \infty .$

$\Rightarrow$ The measure ρ is defined, and

$$\rho\,(L^2(\mathbb{T})) = \int \exp\left(- \textstyle\int_{\mathbb{T}} |v|^6(x)\, dx\right) d\mu(v)$$
$$\leq 1.$$

What about the flow?

Def.: (Sobolev spaces) Let $\sigma \in \mathbb{R}$. We say that

$$f \in H^\sigma \text{ if } \sum_{n \in \mathbb{Z}} \langle n \rangle^{2\sigma} |\hat{f}(n)|^2 < \infty .$$

Fact: NLS is locally well-posed in H^σ for $\sigma > 0$. This means that for $X = H^\sigma \cup \{\infty\}$, $\exists\ \Phi : \mathbb{R}_{\geq 0} \times X \to X$ cont. in both variables. Moreover, $\forall\, \sigma > 0$, $\exists\ q_\sigma < \infty$, $c_\sigma > 0$, s.t.

$$\Phi(t, x) = \infty \Rightarrow t > \frac{c_\sigma}{\langle x \rangle^{q_\sigma}} .$$

$\mu(H^{\frac{1}{2}-}) = 1$, $\mu(H^{\frac{1}{2}}) = 0$

Thm. (Bourgain '94) The measure ρ is invariant for NLS

Pf.: Fix $N \in \mathbb{N}$ large.

let $(\Pi_N f)^\wedge(n) = \hat{f}(n) \mathbb{1}_{|n| \le N}$. (2.4)

Then $\mathrm{Rk}(\Pi_N) = 2N+1 < \infty$.

Consider the equation

$$i \partial_t u_N = -\Delta u_N + \Pi_N (|u_N|^4 u_N) \quad \text{on } R(\Pi_N).$$

This is Hamiltonian, so the Gibbs measure

$$\rho_N(u_N) = \exp\left(-\frac{1}{6}\int |u_N|^6\right) \exp\left(-\frac{1}{2}\left(\int |\nabla u_N|^2 + \int |u_N|^2\right)\right) d\mathrm{Re}\, u\, d\mathrm{Im}\, u$$

is sub-invariant $= \exp\left(-\frac{1}{6}\int |u_N|^6\right) (\Pi_N)_\# \mu$.

Moreover, $\mathbb{E} \left| \int |u|^6(x) - \int |u_N|^6(x) \right|$

$$\le C\, \mathbb{E}\left[\int |u(x) - u_N(x)| \left(|u(x)|^5 + |u_N(x)|^5\right)\right]$$

$$\le C \int \left(\mathbb{E}|u(x) - u_N(x)|^2\right)^{1/2} \sqrt{\mathbb{E}\left(|u(x)| + |u_N(x)|\right)^{10}}$$

$$\le C \int \left(\mathbb{E}\, |u(x) - u_N(x)|^2\right)^{1/2}$$

$$\le C \sum_{|n|>N} \frac{1}{\langle n \rangle^2} \to 0 \quad \text{as } N \to \infty.$$

Therefore, up to subs., $u_N \to u$ a.e. $\overset{\text{dom. conv.}}{\Rightarrow} \exp\left(-\frac{1}{6}\int |u_N|^6\right)$

$\to \exp\left(-\frac{1}{6}\int |u|^6\right)$

in $L^1(\mu)$. Moreover, Fact $\Phi_N(t, \Pi_N u_0) \to \Phi(t, u_0)$

$\Rightarrow$ ρ is sub.-invariant for Φ. Now, we check that

$$\varepsilon^{-1}\rho\left(\{\Phi(\varepsilon, x) = \infty\}\right) \to 0.$$ (2.5)

We know $\{\Phi(\varepsilon, x) = \infty\} \subseteq \left\{ \varepsilon \geq \frac{c_\sigma}{\langle \|x\|_{H^\sigma}\rangle^{q_\sigma}} \right\}$

$$\subseteq \left\{ \|x\|_{H^\sigma} \geq 2\left(\frac{c_\sigma}{\varepsilon}\right)^{\frac{1}{q_\sigma}} \right\}$$

$$\leq \left(\int \|x\|_{H^\sigma}^{2q_\sigma}\, d\rho(x)\right) \frac{\varepsilon^2}{2^{2q_\sigma} c_\sigma^2}$$

$$\leq \underbrace{\int \|x\|_{H^\sigma}^{2q_\sigma}\, d\mu(x)}_{<\infty} \cdot \frac{\varepsilon^2}{2^{2q_\sigma} c_\sigma^2}$$

$$\leq C_\sigma\, \varepsilon^2.$$

Fact: Optimising, we get

$$\|\Phi(t, u_0)\|_{H^\sigma} \leq C(u_0) \log(2+t)^{\frac{1}{2}} \quad \text{a.s.}$$

Question: how general is this? Cite Sun-Tzvetkov '21, BDNY '22

It shows that for a.e. initial data (according to μ), the flow is GWP. How general is this?

Pb. 1: What if the equation is not Hamiltonian?

e.g. $i\partial_t u = -\Delta u + |u|^4 u + \int |u|^4.$

Pb. 2: How general is μ-a.e. initial data?

On $\mathbb{R}^d$, the Lebesgue measure is "canonical", and every "reasonable" operation preserves a.e.. For instance, E full $\Rightarrow$ λE full, $E+E$ full, $E+x$ full, etc.

Not true in ∞ dimension!

Prop.: Consider $\mu_s \sim \exp\left(-\frac{1}{2}\|u\|_{H^s}^2\right)$. They are all singular wrt each other.

Pf.: We find E_s s.t. $\mu_s(E_s^c) = 0$, but $E_s \cap E_{s'} = \emptyset$.

LLN: $$E_s = \left\{ \lim_{j\to\infty} 2^{-j} \| \Pi_{2^j} (1-\Delta)^{s/2} u \|_{L^2}^2 = 4 \right\}$$

Have that, under μ_s,

$$2^{-j} \| \Pi_{2^j} (1-\Delta)^{s/2} u \|_{L^2}^2 = 2^{-j} \sum_{|n|\le 2^j} \langle n\rangle^{2s} \left|\frac{g_n}{\langle n\rangle^s}\right|^2$$

$$= 2^{-j} \sum_{|n|\le 2^j} |g_n|^2$$

$$\xrightarrow{LLN} 2\, \mathbb{E}|g_n|^2 = 4 \text{ a.e.}$$

Therefore, $\mu_s(E_s) = 1$. $\square$

Rk.: The same shows that for $\lambda \neq 1$,

$$\mu_s^\lambda \sim \exp\left(-\frac{\lambda}{2} \|u\|_{H^s}\right) \perp \mu_s \quad !$$

This because $\mu_s^\lambda = \text{Law}\left(\frac{1}{\sqrt{\lambda}} \sum \frac{g_n}{\langle n\rangle^s} e^{in\cdot x}\right)$,

so $$\lim_{j\to\infty} 2^{-j} \| \Pi_{2^j} (1-\Delta)^{s/2} u \|_{L^2}^2 = \frac{4}{\lambda} \neq 4.$$

Question: What happens outside invariance?

Def.: Sub Quasi-Invariance

Let Φ be a Borel flow. We say that a measure μ on $X \setminus \{\infty\}$ is sub-quasi-invariant under Φ if $\forall t \geq 0$,

$$\mathbb{1}_{\{X \setminus \{\infty\}\}}\left((\Phi(t, \cdot))_{\#}\mu\right) \ll \mu.$$

• Equivalently, $\forall E$ s.t. $\mu(E) = 0$, $E \subseteq X \setminus \{\infty\}$,

$$\mu(\{x : \Phi(t, x) \in E\} = 0).$$

• Equiv.: $\exists f_t : X \setminus \{\infty\} \to \mathbb{R}$ Borel s.t.

$\forall F : X \to \mathbb{R}$ meas., $F(\infty) = 0$,

$$\int F(\Phi(t, x))\, d\mu(x) = \int f_t(x) F(x)\, d\mu(x).$$

Thm: Bourgain's invariant measure argument.

Suppose that μ is sub-quasi-invariant for Φ, let f_t as above. Suppose that $\exists\ 1 < p \leq \infty$ s.t.

- $t \mapsto \|f_t\|_{L^p(\mu)}$ is locally bounded
- $\lim_{\varepsilon \to 0} \varepsilon^{-\frac{p-1}{p}} \mu(\Phi(\varepsilon, x) = \infty) = 0$ as $\varepsilon \to 0$.

Then $\forall t \geq 0$, $\mu(\Phi(t, x) = \infty) = 0$.

Pf.: $\mu(\{\Phi(t + \tau, x) = \infty\})$

$$= \mu(\{\Phi(t, x) = \infty\}) + \mu(\{\Phi(t, x) \neq \infty\} \cap \{\Phi(\tau, \Phi(t, x)) = \infty\})$$

$$= \mu\left(\{\Phi(t,x)=\infty\}\right) + \int f_t(x)\, \mathbb{1}_{\{\Phi(\tau,\cdot)=\infty\}}(x)\, d\mu(x)$$

$$\leq \mu\left(\{\Phi(t,x)=\infty\}\right) + \|f_t\|_{L^p} \left(\mu\left(\{\Phi(\tau,\cdot)=\infty\}\right)\right)^{\frac{p-1}{p}} \quad (3.2)$$

So, inductively

$$\mu\left(\{\Phi(t,x)=\infty\}\right) \leq \lim_{n\to\infty} \sum_{j=0}^{n-1} \|f_{\frac{jt}{n}}\|_{L^p}\, \mu\left(\{\Phi(\tfrac{t}{n},\cdot)=\infty\}\right)^{\frac{p-1}{p}}$$

$$\leq \lim_{n\to\infty} \left(\sup_{t\in[0,t]} \|f_t\|_{L^p}\right)\left(n^{\frac{p}{p-1}}\, \mu\left(\{\Phi(\tfrac{t}{n},\cdot)=\infty\}\right)\right)^{\frac{p-1}{p}}$$

$= 0$ by Ass. $\square$

How to show anything about the density?

Go back to ODEs.

$\dot{y} = b(y)$,

and suppose that $b \in C_b^\infty$. Let $\mu_t = \Phi(t,\cdot)_\# \mu_0$. Then

$$\partial_t \mu_t = -\operatorname{div}(b\mu_t). \qquad \text{(Liouville's equation)}$$

If $\mu_0 = \exp(-E(x))\,dx$, ansatz $\mu_t = f_t\mu_0$, and

$$(\partial_t f_t)\mu_0 = \partial_t(f_t\mu_0) = -\operatorname{div}(b f_t \mu_0)$$

$$= -\operatorname{div}(b) f_t\mu_0 - (b\cdot\nabla f_t)\mu_0 - (b\cdot\nabla\mu_0) f_t$$

$$= \left((b\cdot\nabla E) - \operatorname{div}(b)\right) f_t\mu_0 - (b\cdot\nabla f_t)\mu_0 =: Q f_t \mu_0 - (b\cdot\nabla f_t)\mu_0$$

$$\Rightarrow \partial_t f_t = -b\cdot\nabla f_t + Q f_t \quad \leftarrow \text{transport equation}$$

New variable: $\log\left(f_t \circ \Phi(t,\cdot)\right)$ (3.3)

$$\partial_t \log\left(f_t \circ \Phi(t,\cdot)\right) = \frac{(\partial_t f_t)\circ\Phi_t + (\nabla f_t)\circ\Phi_t}{f_t \circ \Phi_t}$$
$$= Q \circ \Phi_t .$$

Then

$$f_t \circ \Phi(t,\cdot)(x) = \exp\left(\int_0^t Q \circ \Phi(s,x)\, ds \right)$$
$$\overset{\text{if } \operatorname{div}(b)=0}{=} \exp\left(E(\Phi(t,x)) - E(x) \right).$$

Jensen-type argument:

$$\| f_t \|_{L^p}^p$$
$$= \int f_t^{p-1} \cdot f_t \, d\mu$$
$$= \int f_t^{p-1} \circ \Phi(t,\cdot)\, d\mu$$
$$= \int \exp\left((p-1) \int_0^t Q \circ \Phi(s,\cdot)\, ds \right) d\mu$$
$$= \int \exp\left(\sum_{j=0}^{n-1} (p-1) \int_{\frac{jt}{n}}^{\frac{(j+1)t}{n}} Q \circ \Phi(s,\cdot)\, ds \right) d\mu$$
$$= \int \exp\left(\frac{1}{n} \sum_{j=0}^{n-1} (p-1) n \left(\int_0^{\frac{t}{n}} Q \circ \Phi(s,\cdot)\, ds \right) \circ \Phi_{\frac{jt}{n}} \right) d\mu$$

Jensen

$$\leq \frac{1}{n}\sum_{j=0}^{n-1}\int \exp\left((p-1)n\int_0^{t/n} Q\circ\Phi(s,\cdot)\,ds\right)\circ\Phi_{j\frac{t}{n}}\,d\mu$$

(3.4)

$$\leq \frac{1}{n}\sum_{j=0}^{n-1}\int \exp\left(n(p-1)\int_0^{t/n} Q\circ\Phi(s,\cdot)\,ds\right) f_{j\frac{t}{n}}\,d\mu$$

$$\leq \sup_{t'\leq t}\int \exp\left((p-1)\frac{t}{\tilde{T}}\int_0^{\tilde{T}} Q\circ\Phi(s,\cdot)\,ds\right) f_{t'}\,d\mu$$

$$\leq \left(\sup_{t'\leq t}\|f_t\|_{L^p}^p\right)^{\frac{1}{p}}\int \exp\left(\left(\frac{pt}{\tilde{T}}\int_0^{\tilde{T}} Q\circ\Phi(s,\cdot)\,ds\right)_+\right)d\mu$$

<u>Meta-thm</u>: Suppose that $\exists\ x_0\in X\setminus\{\infty\}$ s.t. $\forall R>0$,

- $\forall\ P>0$, $\int \exp(P\,Q_+)\,\mathbb{1}_{B_R(x_0)}\,d\mu<\infty$

or

- $\exists\ \tau(\cup)>0$, $1<p<\infty$, s.t.

1. $\int \exp\left(\left|\sup_{t\leq T}\left(\int_0^t Q\circ\Phi_s\,ds\right)_+\right|^p\right)\mathbb{1}_{B_R(x_0)}\,d\mu<\infty$,

2. $\int \exp\left(\tau^{-\frac{p}{p-1}}\right)\mathbb{1}_{B_R(x_0)}\,d\mu<\infty$.

then μ is sub-quasi-inv. and $t\mapsto\|f_t\mathbb{1}_{E_R}\|_{L^q}$ is loc. bounded $\forall q$

where

$$E_R=\{x\in X\setminus\{\infty\}:\ \Phi(-t',x)\in B_R\ \forall t'\leq t\}.$$

⚠ $\int$ is important! (3.5)

Consider $i\partial_t u = -\Delta u + |u|^2 u$ on $\mathbb{T}^d$.

$$\mu_s \sim \exp\left(-\frac{1}{2}\|u\|_{H^s}^2\right) = \exp\left(-\frac{1}{2}\langle(-\Delta)^s u, u\rangle\right).$$

$$Q = \langle(-\Delta)^s u, -i|u|^2 u\rangle$$

$$= \mathrm{Im}\int[(-\Delta)^s u]|u|^2\bar{u}$$

$$\simeq \mathrm{Im}\left(\sum_{n_1-n_2+n_3-n_4=0}\frac{g_{n_1}}{\langle n_1\rangle^s}\frac{\overline{g_{n_2}}}{\langle n_2\rangle^s}\frac{g_{n_3}}{\langle n_3\rangle^s}\frac{\overline{g_{n_4}}}{\langle n_4\rangle^s}\langle n_4\rangle^{2s}\right)$$

$$= -\frac{1}{4}\sum_{n_1-n_2+n_3-n_4=0}\frac{g_{n_1}}{\langle n_1\rangle^s}\frac{\overline{g_{n_2}}}{\langle n_2\rangle^s}\frac{g_{n_3}}{\langle n_3\rangle^s}\frac{\overline{g_{n_4}}}{\langle n_4\rangle^s}\left(\langle n_1\rangle^{2s}-\langle n_2\rangle^{2s}+\langle n_3\rangle^{2s}-\langle n_4\rangle^{2s}\right)$$

$$\mathbb{E}|Q|^2 \sim \sum_{n_1-n_2+n_3-n_4=0}\frac{\left(\langle n_1\rangle^{2s}-\langle n_2\rangle^{2s}+\langle n_3\rangle^{2s}-\langle n_4\rangle^{2s}\right)^2}{\langle n_1\rangle^{2s}\langle n_2\rangle^{2s}\langle n_3\rangle^{2s}\langle n_4\rangle^{2s}}.$$

Consider $|n_1|, |n_2| \sim L$, $|n_3| \sim |n_4| \sim N$, $L \ll N$. Then

almost
$$\gtrsim \sum_{\substack{|n_1|,|n_2|\lesssim L\\ |n_3|\lesssim N}}\frac{(N^{2s-1}L)^2}{L^{2s}L^{2s}N^{4s}} \gtrsim \frac{N^d L^{2d} \cancel{N^{4s}}N^{-2}L^2}{L^{4s}\cancel{N^{4s}}}$$

$$\gtrsim N^{d-2}L^{2d+2-4s}.$$

Therefore, for $d\geq 2$, $\mathbb{E}|Q|^2 = \infty$, but quasi-inv. holds for every $s>2$, $s\in\mathbb{Z}$.

Thm: Qualitative / Quantitative (F. T. / Km.)

Talk about Sun - Tolomeo - Tretikov